Rushes and Opportunities in Entrepreneurship and Cities*

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Abstract
Rushes are a fundamental characteristic of the growth of many industries and cities. To explain these rushes, and better understand the mechanisms of growth, this paper develops a model centered on a new tradeoff between fundamentals and opportunities. Early growth in industries and cities depend critically on the opportunities they provide; whether from entrepreneurship human capital accumulation or land in cities. This analysis provides a blueprint for evaluating policies aimed at encouraging growth. These policies are particularly important in developing countries that are experiencing rapid urban growth both in current cities and the number of new cities.

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1 Introduction

Entrepreneurs leave established firms to start new enterprises that cannot guarantee current, or future, benefits will match those in established firms. Similarly, pioneers leave established cities to develop new cities that lack the benefits of agglomeration. Yet, we observe new firms and new cities being started. Further, rushes of new firms into an industry or people into a city define an important growth phenomenon that is difficult to reconcile with the smaller benefits typically provided in new firms and cities.

The most famous example is the gold rush in California, which turned the 200 person town of San Francisco into a boomtown of 36,000 in just four years. Most recently, the oil boom increased the population of Williston, North Dakota by 42 percent in three years. Similar rushes occur in industries, such as the railroad boom of the 1870s and tech boom of the 1990s. In each of these examples, individuals rushed to take advantage of opportunities that did not exist in established firms and cities. In contrast to the benefits offered by established firms and cities, opportunities offered by a new firm or city depend on when an individual arrives. Early employees in a firm often are granted equity stakes in the firm, and the first residents in a city are able to claim the best land. These opportunities are generally larger for early movers however; it may be beneficial to be early but not too early; for example Douglas Aircraft, Netscape, and Commodore computers.

The objective of this paper is to understand how opportunities shape the creation and growth of new firms and cities. These opportunities influence whether an industry or city experiences a rush, if they do, the size of the rush and the rate of growth.

To do this, I develop a general model of benefits and opportunities that depend on the number of people currently working in a location; for example industry or city. Benefits in a location experience positive and negative spillovers from population. Initially, the positive spillovers outweigh the negative, but eventually the benefits become congested and the benefits begin to decline with population. This causes benefits in a location to have an inverted-U shape with respect to population. This ensures that there are benefits for individuals to concentrate in locations and that it is not efficient to concentrate all individuals in one location. Opportunities may in principal take many forms. However, not all forms of opportunities lead to creation or growth.

To investigate how different forms of opportunities cause different growth patterns; I create a model in continuous time where individuals, at any moment, are able to move from an established location and create, or join, a new location. Total population grows at an
exogenous rate, which is allowed to be nonmonotonic and known by all individuals, where all population is assumed to begin in the established location. Eventually established locations start to experience decreasing returns with population, which increases the relative prospect of creating a new location. A new location can be created by a sole speculator, or a rush. Once a new location is created, individuals continue to move to the new location balancing the benefits and opportunities each location provides. Eventually, new entrants to the new location cease to gain any opportunities and the benefits offered in both locations converge. At this point, the new location becomes an established location, continuing to grow in steady state such that the benefits across established locations are equal.

Only locations with opportunities that are decreasing, or initially increasing but eventually decreasing, with respect to population are created in equilibrium. For intuition consider the case where opportunities are always increasing with population and an individuals is suppose to move at time $t$. This cannot be an equilibrium because this person could postpone her move until time $t + \varepsilon$ and benefit from staying in the established location longer and gaining more opportunities when she moved to the new location.

Cities are formed by rushes if and only if a location’s opportunities are initially increasing but eventually decreasing. Individuals that rush expect to receive the average opportunity of those that rush. A location with initially increasing opportunities cannot be created by a single speculator for a similar rationale as to why a location with opportunities that are always increasing with population cannot be created in equilibrium. In this case, a location is created with a rush, with a determinant size given by the point where the average and marginal opportunity intersect. This implies the size of the rush is fully determined by the form of opportunities.

The initial growth of a location also depends on the form of opportunities it provides. For example, new locations with opportunities that deteriorate slower with population grow faster in equilibrium. The difference in opportunities between being the $k$th person and the $k + \varepsilon$th person determines the amount of benefit an individual is willing to forgo by moving to the new location earlier. The time between when individuals move shrinks as the difference in opportunities shrink. In the extreme case where all opportunities are constant with respect to population, a new location cannot experience slow growth and instead its steady state population moves together in an instant. This case is inconsistent with the observed growth of industries and cities but is the outcome in models without opportunities (Anas, 1992).
The results from the model have implications for policies aimed at encouraging entrepreneurship and urban growth. To demonstrate this, I produce two microfounded models that apply the insights from the model to the context of entrepreneurship and cities.

Consider the incentives for an individual deciding whether to stay in an established industry or move to a new industry. The wages are higher in the established industry due to agglomeration benefits however; the new industry provides unique opportunities to gain experiences. Specifically, new industries have fewer senior workers allowing early movers to advance in their careers. These advancements provide individuals with experiences that build entrepreneurship human capital that is necessary to becoming an entrepreneur. Individuals are willing to forgo the higher wage in the established industry for the opportunity to gain entrepreneurship human capital.

In context of entrepreneurship, the model implies for an industry to be created the entrepreneur human capital opportunities must outweigh the risk in the industry for early movers. The entrepreneur human capital opportunities increase the earlier an individual moves to the new industry and the faster the industry grows in equilibrium, allowing early movers to become relatively more senior faster. This also implies that there should be more entrepreneurs in younger countries. Second, the industry will experience a rush if, and only if, the risk in the industry is sufficiently large that there is an advantage to being early, but not too early. Finally, industries grow faster when the opportunities for later entrants decrease gradually. In example, an industry consisting of many smaller firms will provide entrepreneur human capital opportunities to more entrants implying there should be more entrepreneurs in industries and areas with younger firms.

These implications are consistent with recent empirical evidence on entrepreneurship. The first implication is consistent with evidence from Lazear et al. (2014) that find there is a 40 percent increase in the mean rate of entrepreneurship associated with lowering the median age by one standard deviation, and this holds for each age cohort. For example, amongst thirty year olds, there are more entrepreneurs in Chile, which has a relatively young population, than Denmark, which has a relatively old population. The implication that an area with young firms leads to more entrepreneurship is consistent with the observed clusters of entrepreneurship and classic entrepreneurship intuition by Saxenian (1996), Chinitz (1961), and Jacobs (1970). This implication is also consistent with empirical evidence by Glaeser et al. (2010) that finds initial establishment size is correlated with subsequent employment.
growth due to startups.\footnote{Glaeser et al. (2010) attributes the correlation between initial establishment size and subsequent startup growth to lower costs from being an entrepreneur but the evidence is also consistent with the faster entrepreneurship human capital accumulation suggested in this paper.}

In the model with cities, individuals balance higher wages in the established city, due to agglomeration benefits, and opportunities in the form of land. As individuals move to a new city they get to choose a parcel of land, which differ in their distance to the center of the city and area. The opportunities within a city change as the quality of land changes. Historically, growth patterns have depended on differences in land, even causing some rushes. Although modeling the opportunities a city offers solely as land is simplistic, the model is consistent with recent empirical evidence, historical case studies of cities, and current growth issues in developing countries.

In the model, cities grow according to a life-cycle with three states; before the city is created, an opportunity-dependent growth period, and steady state growth. The opportunity-dependent growth period matches the evidence that cities tend to grow in sequence, each experiencing a period of accelerated growth one after another (Cuberes, 2011). Afterwards, in the model, cities experience steady state growth where they continue to grow at different rates depending on their income functions, matching evidence on the growth of established cities (Black and Henderson, 2003; Henderson and Wang, 2005; Dobkins and Ioannides, 1999; Eaton and Eckstein, 1997; Duranton, 2007).

The implications from the city model are also consistent with the anecdotal evidence on city creation and growth. For a historical example consider Lexington and Louisville Kentucky. These cities were formed 2 years and 75 miles apart from each other. The land around Lexington is not on navigable water but provides ample, and homogenous, farm land in the blue grass region of Kentucky. In contrast, Louisville is positioned directly next to the rapids on the Ohio River, providing large opportunities to early settlers to setup shops on the heavily trafficked river. The surrounding land differed greatly in its distance from the rapids and suitability to build, due to the abundance of swamp land.

The model predicts that Lexington will initially grow faster than Louisville, experiencing a rush, because the land is more homogenous. However, eventually Louisville will grow larger given that it has more natural advantages. Lexington did experience a rush at creation, and by 1790, only eight years after its founding, it had a population of 18,410. In comparison, Louisville’s population was only 200, despite being chartered two years earlier than Lexington. Roughly sixty years later Louisville’s population surpassed Lexington, presumably
when these cities had both entered steady state growth.

The implications from this model provide insights into urban growth initiatives in developing and urbanizing countries. Many countries are trying to balance urban growth in existing cities and encouraging growth in new cities such as Appolonia in Ghana, Tatu in Kenya, and Nova Cidade de Kilamba in Angola. To spur growth these countries have given large opportunities to a select few builders (Cain, 2014). The model predicts this will lead to excessive speculation by a few individuals, and will cause the city to remain mostly empty for an extended duration. This seems to be the case for Nova Cidade de Kilamba, which has seen $3.5 billion in development with a capacity to house half a million people, but half a decade after being started it is described as a ghost town housing only 20,000 inhabitants (Cain, 2014).

This paper demonstrates policies aimed at spurring growth in new cities need to focus on providing broad opportunities in cities. The general model provides a framework for future empirical work that investigates the types of policies that are successful at spurring growth. Future empirical work will be able to study the large scale policy initiatives such as INFONAVIT in Mexico, which aims to transition Mexico’s housing model.

Related Literature

The importance of acquiring human capital through on-the-job training has long been understood since Becker’s seminal work (e.g., Becker (1962, 1975)). The entrepreneurship human capital, developed in this paper, incorporates the benefits of human capital accumulation to explain observed differences in wages. Unpaid internships are an extreme example where individuals forgo higher wages in exchange for opportunities to gain human capital. This mechanism builds on insights in the entrepreneurship literature where some individuals decide to become entrepreneurs as a result of a maximization process across several employment opportunities (Parker, 2004; Lazear, 2004; Edward, 2005; Wagner, 2006).

Recent work by Lazear et al. (2014) extends Becker’s work by allowing an individual’s human capital acquisition to depend on the individual’s rank within the firm. Workers with higher ranks are given more responsibility and exposed to more experiences that allow them to acquire human capital faster. I incorporate this insight and demonstrate that the growth of a firm or an industry depends on these opportunities to acquire human capital.\(^2\)

\(^2\)This paper abstracts from the incentive problems of firms to train new hires. A large literature investigates these incentives, typically with some market imperfections due to asymmetric information or imperfect completion (Katz and Ziderman, 1990; Stevens, 1994; Chang and Wang, 1996; Acemoglu, 1997; Acemoglu
The model predicts there will be more entrepreneurs in areas with younger firms, where individuals can gain entrepreneurship human capital. This result is related to a literature that shows that whether an individual decides to become an entrepreneur is influenced by regional factors (Sternberg and Rocha, 2007; Nanda and Sørensen, 2008; Alvarez et al., 2006; Michelacci and Silva, 2007; Stam, 2007).

In addition, this work is related to a large literature that has tried to explain the correlation between economic growth and abundance of small, entrepreneurial firms. This literature suggests that entrepreneurship spurs competition that leads to technological progress or encourages urban growth increasing efficiency (Glaeser et al., 1992; Acs and Armington, 2006; Glaeser, 2007). This literature also investigates why there are clusters of entrepreneurs. One theory suggests that entrepreneurs may be clustered due to historical reasons that cause differences in human capital or that some locations have a culture of entrepreneurship. Other theories suggest clusters of inputs to entrepreneurship or large customer bases cause clusters of entrepreneurship. In my model, clusters of entrepreneurs are predicted to occur in locations where young individuals can acquire entrepreneurial human capital quickly due to a young population or the abundance of small firms.

This paper also relates to a literature that studies urban growth. These models typically suffer from a coordination problem that leads to large, counterfactual, population swings. Henderson and Venables (2009) are the first to fully characterize a model of urban growth that solves this coordination problem, building on insights from Fujita et al. (1978), Helsley and Strange (1994), Brueckner (2000), and Cuberes (2009). The innovation in Henderson and Venables (2009) is to use durable housing capital that is built by forward-looking builders. The model produces dynamics where cities grow from scratch to a stationary size. From this model they produce numerous important results including implications for the role of government and housing prices.


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4The recent literature builds upon Henderson’s 1974 seminal paper of city sizes and types. Additional work focused on the agglomeration and congestion externalities in cities (Arnott and Stiglitz, 1979; Fujita et al., 1978; Tolley, 1974).
scale due to agglomeration, diseconomies of scale due to congestion, and natural amenities.\textsuperscript{5} Recent working papers by Guner and Eeckhout (2014), Behrens and Robert-Nicoud (2015), Albouy et al. (2015), and Seegert (2015) study the impact of institutions, such as federal taxes and land regulations. In contrast, to these papers that focus on the optimal allocation of population, this paper produces a positive model of urban growth that determines the mechanisms by which individuals decide to create and move to a new city.

The model in this paper provides a general framework that encompasses the insights from this literature. Specifically, the opportunity function is a general tool that solves the coordination problem. The opportunity function can be interpreted as the incentives for builders of durable housing capital, as in Henderson and Venables (2009), incentives for city developers, as in Helsley and Strange (1994), or land as in the urban example in this paper. The use of the opportunity function also allows for cities to continue to grow through time and cities to experience rushes, two empirically relevant features.

The remainder of the paper is structured as follows. Section 2 constructs a general model and provides two microfounded examples. Sections 3 solves the model and reports the key findings through three propositions. Section 4 discusses three applications of the model using the microfounded examples in section 2. Section 5 concludes.

\section{Model}

The objective of the model is to understand how opportunities in new locations shape the creation and growth of these locations. A location could be an industry or a city. This section begins by constructing a general model in section 2.1. Section 2.2 provides two examples of the general model. The first example considers the decision by workers to stay in an established firm or leave the firm to start or join a startup firm. The second example considers the decision by individuals to stay in an established city or become a pioneer that creates or joins a new city. Applications of these examples are discussed in Section 4.

\textsuperscript{5}Other recent urban work includes Behrens et al. (2014), which focuses on the economies of scale due to agglomeration, and Cuberes (2011), which provides empirical evidence on the growth of cities.
2.1 Setup

The economy consists of a continuum of nonatomic homogeneous individuals. Population grows, in continuous time, $t$, according to an exogenous growth rate $\dot{N}(t) = \eta(t) > 0$ that is known by all individuals and allowed to be non-monotonic with respect to time. All individuals begin in an established location, denoted location 1. Each individual decides a time, $\tau$, to leave and start or join a new location, denoted location 2. An individual that never leaves location 1 chooses $\tau = \infty$. Individuals that move to location 2 are given a rank, $k(\tau)$, equal to the measure of population that moved to the location before them.

The population growth of location 1, $\dot{N}_1(t) = \eta(t) - q(t)$, increases with the exogenous growth rate $\eta(t)$, and decreases with the flow of growth to location 2, $\dot{N}_2(t) = q(t)$, which is determined endogenously in the model. For ease of exposition, individuals that move to location 2 are assumed to stay in location 2 forever (in equilibrium, these individuals will not have an incentive to do otherwise). Let $Q(t)$ define the cumulative distribution of individuals that have moved to location 2 at time $t$, $Q(t) = \int_0^t q(t)dt$.

Individuals receive utility from rank independent income, hereafter income, and rank dependent opportunities, hereafter opportunities. The income individuals receive each instant depends on their current location and the population in that location. Income within each location $i$, given by $Y_i(N_i)$, is subject to both economies and diseconomies of scale such that, $Y_i(0) = 0$, $Y'(N) > 0$ for $N < \hat{N}$, and $Y''(N) < 0$ for $N > \hat{N}$. These assumptions ensure that as the population grows there are benefits for individuals to concentrate in locations and that it is not efficient to concentrate all individuals in one location. The opportunities individuals receive depends on their current location and their rank within that location. Opportunities in location 1 are constant and normalized to zero. Opportunities in location 2 are given by the opportunity function $R(k)$, which is continuous and smooth. Finally, to simplify exposition the opportunity function is assumed to be decreasing or initially increasing and then decreasing such that for some rank the average rank is greater than the marginal rank. This regularity assumption is shown in Appendix B to ensure there is an

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6The assumption that population is a continuum of individuals allows for the actual distribution of migration to equal, without uncertainty, the target equilibrium distribution. In addition, the distribution of migration is unchanged by a single individual’s deviation from the targeted equilibrium distribution. This method is very similar to mean field game theory where each individual interacts with the distribution as opposed to each individual in the model.

7The limitation of the model to one established and one new location is imposed for convenience and can be easily loosened to the more general case with numerous established and new locations. An example of heterogeneous new locations in the context of cities is given in Seegert (2015).
equilibrium.

An individual that moves to location 2 at time $\tau$ receives income in location 1 until time $\tau$ and receives income in location 2 afterwards, $y_i = Y_i(N_i)$. An individual that moves to location 2 at time $\tau$ also receives a rank, $k(\tau)$, and the corresponding opportunities $R(k)$. The utility an individual receives from moving from location 1 to location 2 at time $\tau$, defined as $M(\tau)$, combines the utility from income and opportunities an individual receives,

$$M(\tau) = \int_0^\tau e^{-rt} y_1 dt + \int_\tau^\infty e^{-rt} y_2 dt + R(k(\tau)).$$ (1)

Section 3 characterizes the solution of the general model. I now discuss two examples of this setting which I then revisit in the application section (section 4).

### 2.2 Examples

This section provides two different examples that are special cases of the above setting. In both examples, rushes are important features of growth. The first example considers the tradeoff workers face between staying in an established industry, or moving to a new industry. The second example considers the tradeoff between staying in an established city or becoming a pioneer by moving to a new city. Both examples are built using standard models.

#### 2.2.1 Example One: Industry Wages and Human Capital Accumulation

This example builds on the insight that the opportunity to become an entrepreneur depends on an individual’s rank in terms of seniority discussed by James Liang, Hui Wang, and Edward P. Lazear (2015). The key insight is that human capital accumulation on the job depends on the individual’s rank within the firm, an extension of Becker’s seminal work on human capital (e.g. Becker, 1962). This example extends this insight to growing industries that provide individuals with an opportunity to be exposed to more experiences, producing entrepreneurship human capital, that ultimately increases their probability of promotion and ability to start a new firm. However, there are costs to entering an industry too soon. For instance, there is increased displacement risk in young industries. For these reasons there are opportunities to being early, but not necessarily first.
The income in an industry includes agglomeration and rivalry externalities,

\[ Y_i(N_i) = N_i^{-\lambda_i} - \xi_i N_i^{-1}, \tag{2} \]

where agglomeration externalities are given by the first term and rivalry externalities are given by the second term, where \( \lambda_i \in (0, 1) \). Income is initially increasing with population and eventually decreasing following the canonical example in Buchanan (1965).\(^8\) The income is initially higher in industry 1 but eventually, when the industry 2 is fully developed, it will also provide higher incomes. However, until industry 2 has a critical mass, workers that choose to work in industry 2 forgo the higher income in industry 1.

Industries also provide opportunities for workers to gain human capital. These opportunities depend on a worker’s rank within an industry, where the opportunities in industry 1 are normalized to zero. Individuals accumulate human capital faster in industries where the share of the workers in the industry below the individual’s rank is larger. In these faster-growing industries, individuals gain experiences that increase their human capital and ultimately allows them to progress faster in their careers and possibly become entrepreneurs.

The share of the workforce in an industry below rank \( k \) is given by the c.d.f

\[ s(k, \rho_i) = \frac{e^{\rho_i k} - 1}{e^{-\rho_i} - 1}, \tag{3} \]

where \( \rho_i \) is the expected steady-state growth rate of industry \( i \).

An individual’s human capital accumulation \( h = H(s(k, \rho_i)) \) is an increasing function, \( H'(\cdot) > 0 \). Although it is unnecessary to specify the specific human capital accumulation function, one useful formulation is given by,

\[ H(s) = \theta s(k, \rho_i) + d \tag{4} \]

where \( \theta s(k, \rho_i) \) is the amount of human capital accumulation due to the rank effect and \( d \) captures all other human capital accumulation.

While individuals that arrive in an industry early accumulate human capital faster, they also incur more displacement risk. For example, if a worker’s firm goes bankrupt and is in a young industry the worker will have fewer job opportunities within the industry, which values

\(^8\)Buchanan (1965) used a similar formula to discuss the agglomeration benefits and congestion of public goods. The agglomeration and rivalry externalities are allowed to be heterogeneous across industries according to \( \xi_i \) and \( \lambda_i \).
their industry-specific human capital. These risks are captured by the function \( v = V(k, \rho_i) \) with a useful but unnecessary formulation given by,

\[
V(k, \rho_i) = -\rho_i e^{-\rho_i k}(k + c_i),
\]

where a low \( c_i \) denotes a risky industry. In this formulation the risks are decreasing and convex for industries with positive growth rates and increasing and concave for industries with negative growth rates.

The opportunities a worker receives from an industry combines the benefits of human capital accumulation and risk due to displacement. Let \( z_i \) denote the expected value of these opportunities where \( z_i = R(h, v) \) can be formulated as,

\[
R(h, v) = h - v = \theta e^{\rho_i k} \left( \frac{1}{e^{-\rho_i} - 1} + d + \rho_i e^{-\rho_i k}(k + c_i) \right).
\]

Figure 1 depicts the opportunity function in equation (6) for industries with different growth rates and risk. An industry with a high growth rate provides workers with more opportunities to progress in their career and a larger benefit for workers to enter early. An industry with more displacement risk provides fewer opportunities and a smaller benefit to enter early. The analysis in Section 3 characterizes the growth in these different industries based on the shape of these opportunity functions, with applications discussed in Section 4.

Figure 1: Industry Opportunities With Growth And Risk

A) Differences in Growth Rates

B) Differences in Risk
2.2.2 Example Two: Agglomeration Economies and Spiral Land

This example builds on Henderson’s (1974) canonical model of cities. The key insight is that cities combine agglomeration and congestion externalities such that the benefit of living in a city is initially increasing with population but eventually decreasing. Recent work has extended this model to consider the creation and growth of new cities. For example, Henderson and Venables (2009) builds a dynamic model with durable housing capital that avoids population swings in cities that arise in other urban models. The general model builds on these insights and generalizes the role of housing capital as opportunities that exist in cities. The following example provides a simple formulation of the opportunities in cities using insights from the monocentric city models of Alonso (1964), Mills (1967) and Muth (1969).

Each individual $h$ supplies an exogenously fixed $L_h$ units of labor to the production of the composite good $x$. All production occurs in the central business district (CBD). The production of the composite good exhibits constant returns to scale with respect to labor and is subject to city-wide scale agglomeration as a function of city population $N_i$. Because production exhibits constant returns to scale with respect to labor each individual, without loss of generality, can be considered as owning his or her own firm with the production function $f_h(L_h) = A_i N_i^\alpha L_h$. Production amenities in cities are captured by the parameter $A_i$ and the agglomeration economies are given by the term $N_i^\alpha$ where $\alpha \in [0,1]$ captures learning spill-overs, better matching, or sharing of intermediaries.\(^9\)

Production of the composite good also produces pollution, $P(N_i) = (F_i(N_i))^{\psi} p$, where $F_i(N_i)$ is the total production in city $i$ and $\psi > 1$ and $p > 0$ are parameters.\(^10\) Each individual is assumed to bear the average cost of the pollution produced within the city such that every individual receives income,

$$Y_i(N_i) = A_i N_i^\alpha - B_i N_i^\beta, \quad (7)$$

where $\beta = \alpha \psi$ and $B_i = p A_i^\psi$. This micro-foundation of income ensures income is zero with no inhabitants and then strictly rises and falls with population, consistent with the inverted-


\(^10\)This parametric example of pollution is consistent with Tolley (1974)’s description, “The nature of pollution and congestion is that extra pollutants and vehicles do not shift production functions at all at low amounts, and extra amounts have increasingly severe effects as levels are raised until ultimately fumes kill and there are so many vehicles that traffic cannot move.”
U shape in Henderson (1974). Further, the income function is allowed to be heterogeneous across cities according to the parameter $A_i$.

Initially, the income in city 1 is larger than city 2. Individuals that initially move to city 2 forgo the higher income in city 1, until city 2’s population is large enough that the income in both cities are equal. Individuals are willing to move to city 2 because it provides opportunities that city 1 does not. The opportunities in city 2 depends on how many people moved to the city before them.

Specifically, opportunities in city 2 are modeled as parcels of land that individuals claim as they enter the city. This example builds on the monocentric city model pioneered by Alonso (1964), Mills (1967) and Muth (1969). The first person that claims a parcel of land defines the center of the city, where all production occurs. All subsequent entrants claim a parcel that is adjacent, and next in order, to the previously claimed parcel, where parcels are arranged in a spiral.\footnote{The urban economics literature commonly assumes cities grow as a disc from its center, sometimes as concentric circles. The use of a spiral allows the parcels to differ continuously, but the model could easily be rewritten in terms of concentric circles.}

Parcels of land differ in their area, $\nu(k)$, and distance from the center, $\rho(k)$. The benefit individuals derive from the area of their parcel of land is given by $a_i(\nu(k)) = v_i(0.5 \ast (2\pi - 1 + \nu(k)/\pi)^{\gamma_i})$, where $v_i > 0$ and $\gamma_i > 0$ are parameters that can differ across cities. The benefit is allowed to be nonlinear and is normalized to zero for $k = 0$. Individuals also value the distance of their parcel to the center of the city because they incur a cost of commuting, $c_i(\rho(k)) = m_i\rho(k)^{\phi_i}$, where $m_i > 0$ and $\phi_i > 0$ are parameters. The cost of commuting is allowed to be nonlinear with respect to distance, capturing possible fixed costs. The distance and area of the $k$th parcel can be derived as a function of rank $k$ using a discrete analog depicted in Figure 2.\footnote{The spiral is given by an Archimedean spiral with a radius $\rho(\theta) = \theta$ where $\theta$ is a given angle and the lines are defined by a constant angle $\bar{\theta} = 1$, which implicitly assumes there are $2\pi$ parcels of land for a given rotation. The area of parcel $k$ is $\nu(k) = (2k + 1 - 2\pi)\pi$, where $\pi$ is the mathematical constant and this expression is found by integrating between the two curves $\theta$ and $(\theta - 2\pi)$ in polar coordinates between the angles $\bar{\theta}k$ and $\bar{\theta}(k+1)$. Production occurs in the center of the city in the central business district (CBD) and uses the first $2\pi/\bar{\theta} + 1$ parcels of land. The distance of the parcel from the center of the city is given by the radius $\rho(k) = \bar{\theta}k$, which is the closest point in the parcel to the city center. The continuous function is the limit of the discrete analog as the angles that separate parcels of land go to zero $\bar{\theta} \to 0$. The full derivation of distance and area of each parcel is given in Appendix H.}

The opportunity function is the total utility individuals derive from the area and distance
of their parcel of land,

\[ R_i(k) = a_i(k) - c_i(k) = v_i k^{\gamma_i} - m_i k^{\phi_i}. \] (8)

The parameters \( v_i, \gamma_i, m_i, \) and \( \phi_i \) determine the opportunities from land in a given city by allowing the value of land across and within cities to be heterogenous. For example, land in city \( i \) is more heterogeneous than city \( j \) if \( m_i > m_j \). In this case, the slope of the opportunity function is steeper in city \( i \) than city \( j \) capturing the characteristic that the difference in value of parcel \( k \) relative to parcel \( k + \varepsilon \) is greater in city \( i \).

The opportunity function in equation (8) is characterized by one of four cases: i) monotonically increasing, ii) monotonically decreasing, iii) initially increasing and then decreasing, or iv) initially decreasing and then increasing. Different cases cause cities to be created either by a rush, slow migration, or not at all.

Figure 2: Parcels of Land in a City

3 Results

This section characterizes the solution of the general model. Section 3.1 defines the equilibrium and the general set of assumptions. Section 3.2 defines the conditions under which the
new location experiences a rush. Section 3.3 proves the equilibrium uniquely exists. Finally, section 3.4 derives the solution to the growth of location 2 and characterizes the growth in terms of the income and opportunity functions.

3.1 Preliminaries

The model defines a game. A player’s strategy $\tau \in [0, \infty)$ defines the time the player moves from location 1 to location 2. Players receive payoffs according to equation (1) which depends on when the player moves to location 2 and the distribution of when other players move to location 2.

I look for mixed strategy Nash equilibrium according to the CDF $Q(t)$. I restrict the search to symmetric mixed strategy Nash equilibrium, which is without loss of generality because the equilibrium is fully characterized by the CDF, $Q(t)$. The CDF $Q(t)$ does not need to be continuous, allowing for atoms.

**PROPOSITION 1** At any moment in time either both locations are inhabited or one is inhabited and the other is empty.

Before location 2 is created, $t < \tau$, location 2 is empty and no one moves to location 2, $Q(t) = 0$. After location 2 is created, $t \in [\tau, \infty)$, location 2 has strictly positive growth, $Q(t) > 0$. After location 2 is created individuals trade off opportunities in location 2 with higher income in location 1.

3.2 Rushes

The equilibrium may exhibit a rush, defined as an atom in $Q(t)$. In this case, individuals that move at the same time are randomly given a rank and they all have rational expectations of what rank they will be given. Proposition 2 determines the set of opportunity functions that lead equilibrium growth to exhibit a rush at creation.

**PROPOSITION 2** A new location is formed by a rush if and only if the opportunity function is non-monotonic and initially increasing.

The proof of proposition 2 is given in the Appendix C, but the intuition follows from Figure 3. In equilibrium, slow growth to location 2 does not occur when the opportunity function is increasing because individuals can profitably deviate by waiting to move and receive more income and opportunities. However, if the opportunities initially increase and
then decrease, as in Figure 3, then there will be a rush defined as an atom in the cumulative distribution function $Q(t)$. The size of the rush is uniquely determined by ensuring in equilibrium there is no profitable deviation to pre-empt or outlast the rush.

Figure 3 graphs the rank function and average rank function with respect to rank $k$. Individuals that move as part of the rush expect to receive the average opportunity. Define three points; $J$, $B$, and $C$. Point $J$ defines the point at which the average opportunity equals the marginal. Point $B$, defines the point at which the opportunity function’s derivative switches signs and Point $C > 1$ defines the point at which the opportunities are equal to the opportunity of the first mover.

In equilibrium, an individual must be indifferent between being in a rush and moving right after a rush. This condition uniquely determines the size of the rush as the point where the expected value of being in a rush, given by the average opportunity, intersects the opportunity function, which is defined as Point $J$. If the rank function is initially increasing, as in Figure 3, the average opportunity at any point $h$ less than $B$ is less than the opportunity at point $h$. In contrast, at point $C$ the average opportunity is greater than the opportunity at point $C$. Therefore by continuity, point $J$ exists and is greater than point $B$ and less than point $C$.

In an equilibrium with a rush, individuals must not have an incentive to pre-empt the rush, ensuring they receive the opportunity of the first mover. This implies that the expected opportunity of an individual that rushes must be greater than the opportunity of being the first individual in location 2. By continuity, the opportunities at $J$ are greater than the opportunities at $C$, which is the opportunity of pre-empting the rush. Therefore, there is no profitable deviation to pre-empt the rush.

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13The regularity assumption is that the opportunity function decreases enough such that at some point the average opportunity is greater than the marginal.

14A rush is not an equilibrium if the opportunity function is initially decreasing and then increasing. In this case there is an incentive for individuals to pre-empt the rush of migration since the opportunity at the point at which the opportunity and average opportunity intersect is less than the opportunity of being the first mover.
Locations may also experience a rush if the difference in income functions is nonmonotonic. This rush occurs at time $\bar{\tau} > \tau$, which occurs after location 2 is created. The rationale is similar (though in reverse) to the rationale for Proposition 2. Appendix A characterizes the possible rush after creation.

### 3.3 Existence and Uniqueness

The equilibrium is fully characterized by the CDF $Q(t)$ that implicitly defines a time of creation $\tau$, a pair of potential rushes $(J_1, J_2)$ that may occur at $\tau$ or $\bar{\tau}$. The equilibrium is determined by five no arbitrage conditions, given in Appendix A. These conditions ensure there is no profitable deviation when location 2 is created (condition 1), after location 2 is created (condition 2), during a rush that creates location 2 (condition 3), and during a rush that occurs after creation (conditions 4 and 5). These conditions are required for the existence and uniqueness of the equilibrium (Theorem 1) and to fully characterize the equilibrium in Proposition 3, but not for Propositions 1 and 2. The intuition for the proof of Theorem 1 follows the theorem and the proof is given in Appendix D.

**THEOREM 1** There exists a unique mixed strategy Nash equilibrium.

\[15\] Before a location is created, the model is characterized by a pre-emption game but after a location is created the model transitions into a war of attrition game.
Existence and uniqueness of the equilibrium described in Theorem 1 are shown in four steps. First, the size of a rush is shown to uniquely exist, which may be a single speculator. Second, the creation time of location 2 is shown to uniquely exist using the implied expected payoff from the first step. Third, the mix of times to move is constructed demonstrating its existence and uniqueness. Fourth, it is shown that there does not exist any profitable deviation from the constructed equilibrium. These steps are formally shown in Appendix D.

The proof uses the five no arbitrage conditions, derived in Appendix A, to demonstrate that relative to the equilibrium there exists an early time period when it is inferior to start a new location, there exists a late time period when it is better to start location 2, and that the assumed smoothness creates a unique crossing point at which it is optimal to start location 2. This pattern of finding the unique crossing point is repeated to find the unique size of a rush, the unique time location 2 is created, and the mixture of times to move.

The opportunity function solves a collection action problem. First, without opportunities no one would want to leave the established location and start a new one. Second, the opportunity function ensures the uniqueness of the equilibrium, ruling out implausible equilibria with large rushes where the new location gains a critical mass in an instance. To see this note that, in any equilibrium with such a large rush there is an incentive for individuals to pre-empt or outlast the rush by $\varepsilon$, receiving essentially the same income as the individuals that rush but more opportunities. The following section characterizes the equilibrium and section 4 provides several applications of the model.

3.4 Characterizing Growth

(i) Creation and growth

Before a location is created the location is empty and there is no growth, $t < \tau$. The opportunity dependent period occurs after creation and before steady state, $t \in [\tau, \bar{\tau}]$. The growth during this period is characterized by

$$q(t) = \frac{y_1' \eta + e^{\tau r} R' q'}{y_1' + y_2' - e^{\tau r} (r R'' + R''')}.$$  (9)

which is found by using Leibnitz’s rule to take the derivative of the first no arbitrage condition.

\footnote{The unique equilibrium can be shown to be robust to time perturbations. Appendix E demonstrates the robustness by constructing the actual distribution of move times and allows for time perturbations from the targeted distribution.}
in equation (Appendix A.2) with respect to $\tau$. Growth of location 2 in the opportunity-dependent period ensures movers are indifferent between moving and not moving at any time $t \in [\tau, \bar{\tau}]$. Location 2’s growth in this period depends on the rate at which opportunities decrease and incomes increase with population.

**PROPOSITION 3** The growth rate in the opportunity-dependent growth period, not including rushes, depends on the opportunities in location 2 such that (i) a proportional decrease (a flattening) in the opportunities in location 2 causes its growth rate to increase (ii) a level change in the opportunities in location 2 does not affect its growth rate, and (iii) a level decrease (increase) in the opportunities in location 2 causes location 2 to be created later (earlier).

Proposition 3 follows directly from equation (9). Proposition 3 demonstrates three important characteristics of growth. The first characteristic demonstrates that, the growth rate decreases as the difference in opportunities individuals receive decreases, given by $R'(k)$. At first this result may seem counterintuitive, expecting a large difference in opportunities to encourage growth. However, in equilibrium the differences in opportunities must be offset by differences in the level of income an individual receives. Therefore, the benefit in terms of opportunities from being mover $k$ rather than mover $k + \varepsilon$ must be offset by the fact that to be mover rank $k$ rather than $k + \varepsilon$ an individual must move to location 2 earlier, forgoing the higher level of income in location 1 for a longer period of time.

Consider the extreme case where each individual receives exactly the same level of opportunities regardless of when they move to location 2, a flat opportunity function. In this case there cannot be any period of prolonged slow growth because individuals that move early will always have an arbitrage opportunity by waiting. The equilibrium in this case is characterized by one large rush. While this can be equilibrium if the opportunities are constant and zero, it is counter to the empirical facts for industries and cities, but is the prediction of models without opportunities (Anas, 1992). This result highlights the importance of the opportunity function for generating slow growth.

\[ 17 \text{All of the derivations for this section can be found in Appendix F. While it is simpler to work with equation (9), an analytical solution of the ordinary differential equation from equation (Appendix A.2) can be found when it can be rearranged into the linear ordinary differential equation form } q(t) + h(t)Q(t) = g(t). \]

\[ 18 \text{Anas (1992) provides an early example of a model of city growth in the absence of opportunities. The dynamics in Anas (1992) suggest large swings in population when a new city is formed. This model sparked research by Helsley and Strange (1994), Brueckner (2000), Cuberes (2009), and Henderson and Venables (2009), which included land developers or durable housing capital to solve this coordination problem.} \]
The second characteristic demonstrates that, a level shift in opportunities does not affect the rate of growth but does affect when location 2 is created. For example, a policy that provides a lump sum bonus to the individual that creates location 2, will only cause the first individual to move early as a speculator. The lump sum bonus the speculator receives from creating location 2 will be completely offset by the cost of moving earlier.

\( (ii) \) Life-cycle examples

Figure 4 predicts growth in new locations based on the incomes and opportunities in location 1 and 2. All panels begin with a period where location 2 is empty, \( Q(t) \) is zero, and depicts the creation of location 2 by the point at which \( Q(t) > 0 \) for the first time. When the opportunities in location 2 are monotonically decreasing, location 2 is formed by slow growth, as depicted in Panels A and C. When the opportunities in location 2 are nonmonotonic and initially increasing, location 2 is formed by a rush, as depicted in Panels B and D.

Each panel includes a baseline growth path and two comparative growth paths.\(^\text{19}\) The first comparative growth path depicts growth when the income function is proportionately smaller, denoted as a dotted line. In this case, location 2 is created later because incomes are lower and occurs slower because the slope of the income function is flatter. The second comparative growth path depicts growth when the opportunity function is proportionately smaller, denoted as a dashed line. In this case, location 2 is created later because the opportunity of the first mover is smaller and growth occurs faster because the slope of the opportunity function is flatter.

These comparative growth lines provide a foundation for understanding patterns in the data. For example, consider the recent empirical study by Cuberes (2011) that demonstrates that cities tend to grow in sequence, each experiencing a period of accelerated growth one after another. This pattern is predicted in the model when the slope of the opportunity function in a city is relatively flat, causing it to experience an accelerated growth period. If the income functions across cities are heterogeneous, then the population in a the city that experiences the accelerate growth period can catch up to, or even surpass, the population in existing cities. Appendix G considers more examples of comparative growth paths with income and property taxation.

\(^{\text{19}}\)Section Appendix G provides more discussion on the comparative growths.
4 Applications

This section discusses three applications of the model; entrepreneurship in the context of human capital, a case study of Lexington and Louisville, KY, and urban growth in developing countries.

4.1 Entrepreneurship Opportunity

The example in Section 2.2.1 can be applied to entrepreneurship broadly. The specific example in Section 2.2.1 consists of individuals deciding whether to stay in an established industry offering high wages or work in a new industry for the opportunity to gain human capital. The
The proposed mechanism is that workers in new industries have fewer senior colleagues which allows them to be exposed to experiences that produce entrepreneurship human capital, such as more diverse responsibilities that allow a worker to see the larger picture. This section applies these insights broadly to see if they are consistent with, and can add to, our current understanding of entrepreneurship.

An implication of the entrepreneurship model is that there should be more entrepreneurship in young countries that provide younger workers with more opportunities. Specifically, young workers will get promoted faster in younger countries because there is a smaller proportion of senior workers. This mechanism is consistent with recent empirical evidence by Lazear et al. (2014) that find lowering the median age by one standard deviation leads to a roughly 40 percent increase in the mean rate of entrepreneurship. In addition, this finding holds for each age cohort. For example, there are more entrepreneurs at age thirty in Chile, which has a relatively young population, than in Denmark, which has a relatively old population. This finding may help explain Japan’s “entrepreneurship vacuum” and “lost decade,” and provide insights into corrective policies. For example, the entrepreneurship human capital mechanism suggests policies aimed at promoting entrepreneurship should focus on providing younger workers with opportunities that are otherwise absent in aging economies.

Another implication of the entrepreneurship model is that clusters could form around locations with young firms that provide young workers with more entrepreneurship human capital. The implication that areas with young firms promote entrepreneurship is consistent with classic entrepreneurship intuition in work by Saxenian (1996), Chinitz (1961), and Jacobs (1970); though these studies suggest the mechanism is through lower costs of entrepreneurship due to access to venture capitalists and independent suppliers. The entrepreneurship human capital mechanism is consistent with recent empirical work by Glaeser et al. (2010) that finds average initial establishment size is correlated with subsequent employment growth due to startups. In addition, Glaeser et al. (2010) finds that employment growth due to startups in an industry is aided by smaller establishments in other industries within a city, and employment growth in non-startups do not experience the same growth effects. This evidence is consistent with their explanation of lower costs to being an entrepreneur but is also consistent with the entrepreneurship human capital mechanism in this paper.

The entrepreneurship human capital mechanism may help explain the rush of growth
in the technology industry in Silicon Valley, but also places like Austin, TX, Chapel Hill, NC, and Provo, UT. People rushed into the technology industry taking advantage of the opportunities of faster career growth and human capital accumulation. The model predicts these rushes would be larger in areas with younger populations and less senior workers. Austin, Chapel Hill, and Provo are three areas outside of Silicon Valley that experienced large growth in the technology industry. These areas also have extremely young populations, ranking in the youngest 8 percent of counties. For example, the median age in the United States in 2010 was 36.9, in comparison, the median ages in these three cities were 31.9, 33.1, and 24.6 respectively.

This paper provides a new mechanism for entrepreneurship consistent with recent empirical studies.

4.2 Lexington and Louisville Kentucky

Louisville and Lexington, Kentucky provide an interesting comparison of the creation and growth of cities because they were chartered two years (1780 and 1782) and seventy-five miles apart. Louisville is located next to the falls of the Ohio, which was the only navigational barrier on the Ohio River at the time. The falls provided a convenient stopping point for the steady flow of traffic on the river, suggesting a large level of production amenities, $A_i$. The land surrounding the rapids is heterogeneous both in distance from the river and suitability to build due to excessive swamps. In contrast, Lexington is not on a navigable river. However, Lexington is located in the center of the inner Bluegrass Region, which provides vast amounts of fertile and homogenous land. Anecdotally, land was an important determinant for the growth of these cities, as it is in many contexts of urban growth (Wade, 1996).

The opportunity function that captures the benefits from acquiring a parcel of land differs significantly between Lexington and Louisville. This heterogeneity in land is captured by differences in the slope of the opportunity function. Specifically, the slope of the opportunity function is steeper in Louisville than in Lexington, $m_{Louisville} > m_{Lexington}$. Due to this difference in opportunity functions, the results in Section 3.4 predict that Lexington will experience a rush of migration early on but Louisville with its superior production amenities $A_{Louisville} > A_{Lexington}$ will eventually become larger when both cities reach steady state growth.

These predictions are corroborated by history as shown in Figure 5. Lexington experienced a rush reaching a population of 18,410 by 1790, only eight years after being founded.
In the same year, Louisville’s population was 200, despite being chartered two years earlier than Lexington. It took Louisville roughly sixty years, possibly when these cities reached steady state growth, for it to surpass Lexington in population.

Figure 5: Lexington and Louisville Model Prediction and Actual Growth

A) Lexington

B) Louisville

Note: The model predicts the rush of migration into Lexington and the slow initial growth in Louisville based on the differences in the heterogeneity of land in both cities. The other features could be predicted given other inputs on the opportunity and income functions. The model prediction lines are determined using a best fit line.

4.3 Developing and Urbanizing Countries

The implications from the model provide a framework to assess policy initiatives aimed at urban growth in developing and urbanizing countries. Many countries are experiencing increased urbanization and are coping by expanding existing cities and creating new ones (Henderson and Wang, 2007; Bertrand and Dubresson, 1997). This paper suggests that the rules about land ownership can affect the creation and growth of new cities by affecting the opportunities new residents receive. This is consistent with evidence on the impact of customary land rights in Africa (Magigi and Drescher, 2010; Henderson et al., 2013). Many cities in Sub-Saharan Africa provide employment opportunities that are often used to diversify the economy (Brueckner and Selod, 2009). Recent studies in Mexico and Argentina demonstrate the impact of land rights on investment, migration, and human capital accumulation (Gonzalez-Navarro et al., 2014; Galiani and Schargrodsky, 2006). Land policies have also been found to be important in growth and reducing poverty in India (Besley and Burgess, 2000). Important research by Galor et al. (2009), Rossi-Hansberg and Wright (2007), Black and Henderson (1999), and Henderson and Wang (2005) connects land policies and human capital accumulation that explains differences in opportunities, poverty, and urbanization. For example, the Himo settlement in Kilimanjaro in Sub-Saharan Africa was established on land that is mainly held under customary land rights that limit the city’s ability to expand outward (Magigi and Drescher, 2010).
sify household income and mitigate risk (Ellis and Harris, 2004). The opportunities in these
cities may be minimal, but remain steady for later migrants. In contrast, a relatively new
trend in African urbanization is the creation, from scratch, of new satellite cities: including
Appolonia in Ghana, Tatu in Kenya, and Nova Cidade de Kilamba in Angola. Many polit-
ical detractors fear the opportunities provided by these cities is concentrated in a few first
movers (Cain, 2014).

The model predicts that cities with opportunities that are relatively flat will experience
faster growth than cities that provide large opportunities to a few early movers. In addition,
the model predicts that large opportunities to early movers will encourage a few speculators
to move early, capturing the opportunities, but the city will remain mostly empty for an
extended period. This seems to be the case for Nova Cidade de Kilamba, which is a $3.5
billion development built to house half a million people but half a decade later is typically
described as a ghost town with only 20,000 inhabitants (Cain, 2014).

5 Conclusion

There are large literatures that grapple with the incentives that cause the observed patterns
of entrepreneurship and urban growth. These literatures have produced many explanations
for the observed clusters of entrepreneurship (e.g., clusters of suppliers and demand) and
the sequential growth of cities (e.g., durable housing capital and land developers). Here I
provide a general model of industry and city growth with the novel addition of an opportunity
function that encompasses many of these explanations. The opportunity function allows for
slow growth and rushes; two elements observed in industry and city growth.

I provide separate models of entrepreneurship and city growth building on canonical
models by Becker (1962, 1975) and Henderson (1974). The entrepreneurship growth model
considers the incentives for an individual in an established industry to move to a new in-
dustry. The wages are higher in the established industry but the new industry provides
the opportunity to acquire entrepreneurship human capital. This opportunity drives indus-
trial and entrepreneurship growth in a way that is consistent with empirical evidence on
entrepreneurship by Glaeser et al. (2010) and Lazear et al. (2014).

The urban growth model considers the incentives for an individual in an established
city with a high income to move to a new city with low income. The urban literature has
almost exclusively considered the tradeoff within cities between agglomeration benefits and
congestion that cause the established city to have a higher income. Only recently have papers (e.g., Henderson and Venables (2009)) begun to consider new tradeoffs with forward looking individuals to produce dynamic growth that matches the observed patterns. The opportunity function, developed in this paper, provides a general framework for the incentives that exist for individuals to forgo higher incomes in established cities and start and move to new cities. The resulting growth from the model matches the empirically observed patterns that cities tend to grow sequentially and sometimes experience rushes (Cuberes, 2011).

Rushes are a defining characteristic of many industries and cities. For example, consider San Francisco without the gold rush or tech boom. This paper demonstrates the opportunity function fully characterizes whether a rush will occur and how large the rush will be. However, this paper leaves to future research to answer whether the rushes caused by these opportunity functions occur optimally with the optimal size.

Although the paper treats entrepreneurship and cities separately, they are intricately linked. For example, Henry Ford and Detroit, MI, George Eastman and Rochester, NY, Will Kellogg and Battle Creek, MI, and William Shockley and Mountain View, CA. For a more recent example consider the links between the urban growth and tech booms in cities like Austin, TX, Salt Lake City, UT, and the research triangle in North Carolina. A rush occurred in each of these cities because of a rush in the technology industry. Why did the rushes in the technology industry occur in those cities? The entrepreneurship human capital model suggests the young age distribution in these cities was a contributing factor. The urban model suggests low land and cost of living costs could provide opportunities to young workers and young firms. The general opportunity function encompasses these mechanisms but more focused work is needed on the intersections and nuances of the different types of opportunities that exist.

Similarly, urban policies in developing countries should explore opportunities in housing and land reforms but also entrepreneurship and other forms of human capital. Many countries are quickly urbanizing and facing a tradeoff between expanding existing cities and providing incentives to start new cities. In many countries, such as Ghana and Mexico, land and housing reform will provide new opportunities for growth. However, this paper demonstrates growth will be lackluster if these opportunities are not shared broadly.

21 A recent Forbes article by Joel Kotkin, “The New Places Where America’s Tech Future Is Taking Shape,” asserts Austin, TX, Raleigh, NC, Columbus, OH, Houston, TX, and Salt Lake City, UT all saw double digit rate expansions of technology employment in the last decade despite flat or declining rates in San Francisco, CA, Boston, MA, and San Jose, CA.
The model is sufficiently general and tractable that it can be used as a building block for future models of entrepreneurship and urban growth. Future work may explore different types of opportunity functions and estimate the impact of different opportunities. In addition, future work could study different distortions to the opportunity function, such as income and property taxation, and their effect on the growth of industries and cities. Finally, future work may consider whether the existing opportunities are producing the efficient patterns of growth.
References


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Appendix A  No Arbitrage Equilibrium Conditions

This section describes five no arbitrage conditions not required for Propositions 1-2 but required to prove the existence and uniqueness of the equilibrium (Theorem 1) and to fully characterize the equilibrium in Proposition 3. In the urban context these no arbitrage conditions are typically called spatial equilibrium conditions.

The cumulative distribution function $Q(t)$ is an equilibrium if and only if the following five no arbitrage conditions are met, each referring to a different stage of the development of location 2. The first two conditions are satisfied in any equilibrium at different moments in time. The last three conditions are only relevant in the case with rushes. Rushes occur if there are nonmonotonicities in the opportunity function or in the difference in incomes between two locations. The CDF $Q(t)$ implicitly defines a threshold $\tau$, when location 2 is created and a pair of rushes $(J_1, J_2)$ (that may occur at times $\tau$ or $\bar{\tau}$ respectively).

The five no arbitrage conditions ensure that in the symmetric mixed Nash equilibrium individuals are indifferent regarding when to move to location 2. Specifically, the no arbitrage conditions imply individuals are indifferent i) at the creation of location 2, ii) after location 2 is formed and growing, and if applicable, iii) before and after a rush creates location 2 and iv) and v) before and after a rush at time $\bar{\tau}$, which occurs because of nonmonotonicity in the difference between incomes in the two locations.

(i) Location Creation

Initially the value of creating location 2 is lower than staying in location 1 because the opportunity to be the first person is not large enough to make up for the forgone income from leaving location 1. Income within location 1 decreases with time as the location grows and eventually the opportunity of being the first person is larger than the forgone income in location 1. Given that the income function is smooth and concave this implies that the point at which the payoffs from creating location 2 and staying in location 1 are equal,

$$\int_{\tau}^{\infty} e^{-rt} y_2(N_2) dt + R(J_1) = \int_{\tau}^{\infty} e^{-rt} y_1(N_1) dt,$$

(Appendix A.1)

where $R(J_1) = R(1)$ if location 2 is started without a rush and $R(J_1) = (1/J_1) \int_{0}^{J_1} R(k) dk$ if location 2 is started with a rush. This condition determines the unique time, $\tau$, when location 2 is created. This condition ensures there are no arbitrage opportunities for individuals to pre-empt or postpone the creation of location 2.

(ii) Growth After Location 2 is Formed

This condition ensures that movers receive the same payoff after location 2 is formed $t > \tau$, determined by $\dot{M} = 0$. This condition can be written as,

$$y_1(N_1(\tau)) = y_2(N_2(\tau)) - e^{rt} \frac{\partial R(k(\tau))}{\partial \tau},$$

(Appendix A.2)

by rearranging the derivative, found with Leibniz’s rule with respect to $\tau$, of the payoff in equation (1). The economic interpretation is that income in location 1 must equal the income
in location 2 plus the present value benefit of receiving the opportunities associated with being rank $k$, as opposed to rank $k + \varepsilon$. This condition must hold for all time $t \in \left[ \tau, \infty \right)$.

(iii) Rush at Creation

If location 2 is created by a rush of migration, individuals in the rush expect to receive the average rank payoff. The size of the rush $J_1$ is determined by the no arbitrage condition,

$$\frac{1}{J_1} \int_0^{J_1} R(k) \, dk = R(J_1), \quad \text{(Appendix A.3)}$$

which states that the average rank benefit must equal the marginal rank benefit at the rank equal to the rush size. If this condition did not hold there would be an arbitrage opportunity for individuals in the rush to wait and move right after the rush of migration (if $\bar{R}(J_1) < R(J_1)$) or for individuals after the rush to join the rush (if $R(J) > R(J)$). Finally, there are no arbitrage opportunities for individuals to pre-empt the rush because $\bar{R}(J_1) > R(1)$. This inequality holds because the average benefit is maximized at the point at which the average benefit intersects the marginal benefit.

(iv) Rush After Creation (part 1)

The no arbitrage condition for rushes when the difference in incomes is nonmonotonic has two parts, defining both the timing and the size of the rush. First, individuals must be indifferent between moving immediately before the rush or joining the rush. This no arbitrage condition states that the difference in incomes between the locations is equal to the difference between the opportunity function just before the rush and the average opportunity of individuals in the rush,

$$y_1(N_1(\tau_2)) - y_2(N_2(\tau_2)) = R(k - \varepsilon) - \bar{R}(J_2). \quad \text{(Appendix A.4)}$$

(v) Rush After Creation (part 2)

Second, to ensure there are no opportunities to benefit from waiting until just after the rush rather than rushing (or an individual to rush instead of moving just after the rush) the difference in incomes between the locations must equal $\bar{R}(J_2)$ which is the average opportunity of the individuals that rush,

$$y_1(N_1(\tau_2)) - y_2(N_2(\tau_2)) = \bar{R}(J_2). \quad \text{(Appendix A.5)}$$

Appendix B  Regularity Assumption

Regularity Assumption If the opportunity function $R(k)$ is monotonically increasing or initially decreasing and then increasing there does not exist an equilibrium.

Suppose toward contradiction there is an equilibrium for a location with an opportunity function that is monotonically increasing. Consider a deviation from this equilibrium where an individual moves to location 2 at time $t + \varepsilon$ instead of their equilibrium prescribed time
This is a profitable deviation because the individual receives more income and more opportunities. The income in location 1 is higher than location 2 by definition of location 2 not having reached steady state growth. Therefore the deviation of moving at \( t + \varepsilon \) instead of \( t \) allows the individual to receive the higher income in location 1 for \( \varepsilon \) longer. The individual also benefits from delaying her move and receiving a higher level of opportunities because, by assumption, the opportunities in location 2 are monotonically increasing with rank. Therefore there is no equilibrium in which the opportunity function \( R(k) \) is monotonically increasing.

Suppose toward contradiction there is an equilibrium for a location with an opportunity function that is initially decreasing and then increasing. Consider a deviation from this equilibrium where an individual moves to location 2 at time \( t + \varepsilon \) instead of their equilibrium prescribed time \( t > \tau \), where the opportunity function is increasing in rank. Then by the same reasoning as above the deviation will be profitable as the individual will receive more income and better opportunities.

Appendix C  Proof Proposition 2

PROPOSITION 2  A new location is formed by a rush if and only if the opportunity function is non-monotonic and initially increasing.

PROOF

PROOF STEP 1: NO EQUILIBRIUM WITH SLOW CREATION. Assume toward contradiction that the opportunity function is non-monotonic and initially increasing, as depicted in Figure 1, but there exists an equilibrium without a rush of migration. For this to be an equilibrium there must be no profitable deviation from the equilibrium strategy. Consider the first individual to move to location 2. Income in location 2 is lower than in the existing location but increases with time as more individuals move to location 2. The opportunity function is also initially increasing with time by assumption. This demonstrates that the first individual has an incentive to deviate and move later, avoiding some time with lower income and receiving more opportunities associated with being a lower rank. Therefore, there does not exist an equilibrium where the location is created with gradual migration when the opportunity function is non-monotonic and initially increasing.

PROOF STEP 2: NO RUSH WITH MONOTONIC OPPORTUNITIES. To demonstrate that the creation of a location by a rush of migration is possible only when the opportunity function is non-monotonic and initially increasing, consider the cases with a monotonic opportunity function and a non-monotonic opportunity function that is initially decreasing. Begin with a monotonic opportunity function. The no arbitrage condition for a rush of migration defines the size of the rush by the point at which the average opportunity function intersects the opportunity function. For monotonic opportunity functions the average opportunity function and the opportunity function intersect only when the rank equals one. For a rush larger than one, an arbitrage condition exists because either \( \bar{R}(J) < R(J) \) (when the opportunity function is increasing) or \( \bar{R}(J) > R(J) \) (when the opportunity function is decreasing).
PROOF STEP 3: NO RUSH WITH INITIALLY DECREASING NON-MONOTONIC OPPORTUNITIES. Consider a non-monotonic opportunity function that is initially decreasing, as depicted in Figure 2. In this case there does exist a point $J$ such that the no arbitrage condition in equation (Appendix A.3) holds. However, there exists an arbitrage condition for individuals to pre-empt the rush because $R(1) > \bar{R}(J)$. This inequality holds because $\bar{R}(J)$ is the minimum value for $\bar{R}(k)$, as depicted in Figure 3. Therefore there exists an arbitrage opportunity, and no equilibrium exists.

PROOF STEP 4: EXISTENCE OF EQUILIBRIUM WITH RUSH. To demonstrate that an equilibrium does exist where location 2 is formed by a rush of migration, consider the payoffs of individuals that create the location with a rush of migration. Individuals in a rush of migration expect to receive the average rank payoff. The no arbitrage condition given in equation (Appendix A.1) defines when the location is created, even when created with a rush of migration, where $R(k)$ is replaced by the average within the rush $\bar{R}(J_1) = (1/J_1) \int_0^{J_1} R(k) dk$. The size of the rush $J_1$ is determined by the no arbitrage condition,

$$\frac{1}{J_1} \int_0^{J_1} R(k) dk = R(J_1),$$

which states that the average rank benefit must equal the marginal rank benefit at the rank equal to the rush size. If this condition did not hold there would be an arbitrage opportunity for individuals in the rush to wait and move right after the rush of migration (if $\bar{R}(J_1) < R(J_1)$) or for individuals after the rush to join the rush (if $\bar{R}(J) > R(J)$). Finally, there are no arbitrage opportunities for individuals to pre-empt the rush because $\bar{R}(J_1) > R(1)$. This inequality holds because the average benefit is maximized at the point at which the average benefit intersects the marginal benefit. Therefore there exists an equilibrium where location 2 is formed by a rush when the opportunity function is non-monotonic and initially increasing.

\[\square\]

Appendix D  Proof of Theorem 1: Existence and Uniqueness

This section formalizes the existence and uniqueness of the equilibrium.

The five no arbitrage conditions define the set of possible equilibria with or without rushes. The smoothness of these conditions is found to produce an equilibrium which is unique. Specifically, the no arbitrage conditions imply i) a unique creation time of the new location, ii) a unique distribution of migration after location 2 is formed, and iii) unique rush sizes and times if rushes occur in equilibrium.

Existence and uniqueness of the equilibrium described in Theorem 1 are shown in four steps. First, the size of a rush of migration is shown to uniquely exist. Second, the creation time of the new location is shown to uniquely exist using the implied expected payoff from the first step. Third, the distribution of migration times is constructed demonstrating its
existence and uniqueness. Finally, it is shown that there does not exist any profitable deviation from the constructed equilibrium.

PROOF STEP 1: THE SIZE OF RUSHES. The size of a rush at the creation of location 2 is determined by the point at which the average opportunity in the new location equals the marginal opportunity. In the case where the opportunities are monotonically decreasing there is no rush because the marginal and average intersect only for the first individual. Otherwise, the intersection point is defined by the maximum point of the average opportunities in the new location, which is defined by a unique point. Similarly, there exists a unique time \( \tau \) defined by the fourth and fifth no arbitrage conditions.

PROOF STEP 2: LOCATION CREATION TIME. The first no arbitrage condition given in equation (Appendix A.2) states that the expected payoff of moving at the time of location creation is the same as moving after location 2 is created. Therefore, by the smoothness of the opportunity and income functions, the creation time of the new location, \( \tau \), uniquely exists.

PROOF STEP 3: CONSTRUCTING THE CDF. The CDF can be found by inverting the first no arbitrage condition. The distribution exists and is unique because the income and opportunity functions are both monotonic after the creation of the location and are both sufficiently smooth. The distribution can also be solved by the differential equation given in equation (9) given the boundary points found in step 2.

PROOF STEP 4: NO PROFITABLE DEVIATION. Steps 1 through 3 construct the unique equilibrium. The final step is to demonstrate that there is no profitable deviation to ensure it is an equilibrium. The equilibrium is constructed using the no arbitrage conditions. The first no arbitrage condition ensures there are no profitable deviations from pre-empting or delaying the creation of the new location. The second no arbitrage condition ensures there is no profitable deviation after location 2 is formed. The third no arbitrage condition ensures there are no profitable deviations from pre-empting or delaying a rush at creation. The final two no arbitrage conditions ensure there is no profitable deviation if there is a rush due to nonmonotonicity of the difference in the incomes in the two locations. These no arbitrage conditions exhaust all possible deviations and ensure that no profitable deviation exists.

Theorem 1 asserts the existence and uniqueness of the equilibrium in the focal set of equilibria where slow migration occurs without periods of inaction. Other equilibria may exist such that the location is created with one large rush of migration; however these equilibria are not of interest as they are not empirically important.

Appendix E  Robustness

This section considers the robustness of the Nash equilibrium in the paper to trembles in the timing game. The specific Nash equilibrium refinement considered is a type of trembling hand equilibrium similar to Abreu and Brunnermeier (2003) and defined as a time tremble in Anderson et al. (2013). Individuals decide when to move to location 2, targeting a time
\( \tau \). The time tremble allows for slight perturbations such that the actual time the individual moves is \( \hat{\tau} \), where the difference between the target and actual time \( \tau - \hat{\tau} \) is exponentially distributed with mean \( \epsilon > 0 \). The structure, though not the realization of the random variable, is common knowledge.

**THEOREM 2** With the smoothness of the income and opportunity functions the unique equilibrium \( Q^*(t) \) is an \( \epsilon \)-robust equilibrium.

To demonstrate that an equilibrium is an \( \epsilon \)-robust equilibrium, construct a sequence \( \{Q_\epsilon\} \) of \( \epsilon \)-robust equilibria converging to \( Q^* \) as \( \epsilon \to 0 \). The construction of the sequence of \( \epsilon \)-robust equilibria follows in three steps. First, given the time perturbations find the actual distribution of moving times and the resulting expected payoff from moving within its support. Second, define the no arbitrage condition with time perturbations as the derivative of the expected payoff found in step 1 with respect to the move time \( \tau \). Third, demonstrate that the sequence \( \{Q_\epsilon\} \) can be constructed using the equilibrium \( Q^* \), or a slight perturbation from it, because \( Q^* \) solves the differential equation found in step 2.

**PROOF STEP 1: EXPECTED PAYOFF WITH TIME PERTURBATIONS.** Define the actual distribution when all individuals attempt to move according to \( Q \) as

\[
G_\epsilon(Q)(t) = \int_0^\tau [1 - e^{-(\tau-s)r/\epsilon}]dQ(s) \tag{Appendix E.1}
\]

and the expected payoff of moving at time \( \tau \) as

\[
M_\epsilon(Q)(t) = \int_\tau^\infty e^{-(s-\tau)r/\epsilon}(y_2(G_\epsilon(Q)(s)+R(G_\epsilon(Q)(s)))(1/\epsilon)ds + \int_0^\tau e^{-sr/\epsilon}y_1(\bar{N}-G_\epsilon(Q)(s))ds. \tag{Appendix E.2}
\]

**PROOF STEP 2: NO ARBITRAGE CONDITION WITH TIME PERTURBATIONS.** The no arbitrage condition is found by taking the derivative of the expected payoff from moving with respect to move time \( \tau \), given by

\[
M'_\epsilon(Q)(t) = y_1(\bar{N} - G_\epsilon(Q)(\tau)) - y(G_\epsilon(Q)(\tau)) - R(k(\tau)) - \int_\tau^\infty e^{-rt} \left( - \frac{\partial R(k(\tau))}{\partial \tau} \right) dt = 0 \tag{Appendix E.3}
\]

**PROOF STEP 3: CONSTRUCTING \( \{Q_\epsilon\} \).** Theorem 1 implies there is a unique noiseless equilibrium \( Q^* \) that ensures there are no arbitrage opportunities as defined by equation (Appendix A.2). The no arbitrage condition with time perturbations in equation (Appendix E.3) is a smooth perturbation of the no arbitrage condition in equation (Appendix A.2). Therefore, since \( Q^* \) solves equation (Appendix A.2) it also solves equation (Appendix E.3) and it clearly converges to itself.

\[\text{The convergence norm is the } \text{Levy Metric}, \text{ following Anderson et al. (2013).}\]
Appendix F  Location 2 Growth (Proposition 3)

This section solves for the distribution that defines the growth of location 2.

It is possible to solve for location 2’s growth by taking the derivative of the payoff for a mover in equation (1) with respect to the time a person moves \( \tau \). The second no arbitrage condition ensures in equilibrium there is no profitable deviation from the equilibrium. This implies that the derivative of the payoff to movers with respect to moving time must be zero for all moving times. The derivative is found using Liebnitz’s rule and noting that the rank function is a function of moving time but that income \( y_i \) is a function of time not moving time.

\[
\dot{M} = e^{-r\tau} y_1(N_1(\tau)) - e^{-r\tau} y_2(N_2(\tau)) + \frac{\partial R(k(\tau))}{\partial \tau} = 0
\]

The derivative of this condition with respect to the moving time \( \tau \) produces the condition,

\[
y_1'(\eta - q) = y_2'q - re^{-r\tau} R'q - e^{-r\tau}(R'q' + qR'').
\]

This condition is an ordinary differential equation that can be solved explicitly or rearranged to get \( q(t) \) on one side,

\[
q(t) = \frac{y_1'q + e^{r\tau} R'q'}{y_1' + y_2' - e^{r\tau}(rR'' + R')}.
\]

Appendix G  Policy Effects: Comparative Statics

The objective of this section is to understand how location creation and growth are affected by changes in the income and opportunity functions. The general findings are summarized in comparative statics 1-3. These comparative statics are then applied to the two examples described above in section 2 to demonstrate how different policies affect location growth.

Appendix G.1  The Role of the Opportunity Function with Implications for Property and Income Taxation

This section investigates how changes in the opportunity and income function’s level and slope affect growth and rushes in the context of cities. The effect of property taxation on location growth is demonstrated using comparative static 1. The effects of property taxation on rushes are demonstrated using comparative static 2. Finally the effects of income taxation and the income function is shown using comparative static 3.

Appendix G.1.1  Location Growth

The effect of the opportunity function on migration is found by taking the derivative of migration given in equation (9) with respect to the level \( R_2(k) \) and slope \( R'_2(k) \) of the opportunity function.
COMPARATIVE STATIC 1 The growth rate in the opportunity-dependent growth period, not including rushes, depends on the opportunities in the new location such that (i) a proportional decrease (a flattening) in the opportunities in a location causes the location's growth rate to increase (ii) a level change in the opportunities in the location does not affect its growth rate, and (iii) a level decrease in the opportunities in a location causes the location to be created later.

i) Proportional Shift

As the difference in opportunities individuals receive, \( R'(k) \), decreases, the growth rate of a location increases. At first this result may seem counterintuitive, expecting a large difference in opportunities to encourage growth. However, in equilibrium the differences in opportunities must be offset by differences in the level of income an individual receives. Therefore, the benefit in terms of opportunities from being mover \( k \) rather than mover \( k + \varepsilon \) must be offset by the fact that to be mover rank \( k \) rather than \( k + \varepsilon \) an individual must move to the new location earlier, forgoing the higher level of income in the existing location for a longer period of time.

Consider the extreme case where each individual receives exactly the same level of opportunities regardless of when they move to the new location, a flat opportunity function. In this case there cannot be any period of prolonged slow growth because individuals that move early will always have an arbitrage opportunity by waiting. The equilibrium in this case is characterized by all growth in the opportunity-dependent growth period occurring in one instant, such that all individuals receive the same income. While this is an equilibrium, it is counter to the empirical facts; cities do not grow large and then split into two cities overnight. However, this equilibrium is what is typically assumed in the literature because, by ignoring opportunities offered by new cities, these models implicitly assume a flat opportunity function (Anas, 1992).

ii) Level Shift

As another example, consider the policy of offering the individual that creates location 2 a lump sum bonus from the federal government to encourage growth in new locations. The effect of this policy will be that location 2 is created earlier but the migration times of all individuals other than the creator will remain the same. The bonus the federal government offers will be completely offset, in terms of utility to the individual that creates location 2, by the extended period of time in which the first individual lives in location 2 alone, forgoing the higher income in location 1.

Consider the effect of a level shift in the opportunity individuals receive from moving to location 2. First, from equation (9) it can be seen that a level shift has no affect on the growth rate of a location, since \( q(t) \) does not depend on \( R_2(k) \). Because the opportunity function remained unchanged other than a positive shift, the migration pattern in the opportunity-dependent growth period will remain unchanged, causing a positive level shift in benefits to migrating to location 2. To maintain equilibrium, income in the established location, which will be foregone by the movers, must be greater at the time location 2 is created to counter the increase in opportunities. This implies that location 2 will be formed earlier because
income in the established location decreases as the population increases.

**iii) Example Two Revisited With Property Tax**

Consider the effect of property taxation in the first example. The land value, determined by the area and distance from the city center, determines the opportunity benefits individuals receive from moving to the new city. The opportunity function with property taxation is given by,

\[ R(k) = (1 - \tau_p)(vk^\gamma - mk^\delta). \]  

(Appendix G.1)

where \( \tau_p \) is the property tax rate.

To determine the effect of property taxation on city growth, first consider how property taxes affect the opportunity function and then use the comparative static 1 to determine the effect on city growth. Property taxation has two effects on the opportunity function. First, property taxes decrease the benefit individuals receive when moving to city 2 because some of the benefit is taxed away. Second, property taxation decreases the difference in benefits received by an individual with rank \( k \) and an individual with rank \( k + \varepsilon \). The effect of property taxation on the opportunity function is depicted in Figure 6 where city B has a larger property tax than city A.

Using the result in comparative static 1, higher property taxes cause cities to be created later but grow faster. City 2 is created later because property taxes decrease the level of opportunities in the city. The property tax also flattens the opportunities causing the city to grow faster. Figure 4 depicts a baseline growth path, solid line, and a growth path with the property tax, long dashed line. In each of the four panels the growth path with the property tax has the city created later in time but with a faster growth path than the baseline.

\(^{23}\)Formally, this is given by \( \frac{\partial R(k)}{\partial \tau_p} > 0 \), where \( \frac{\partial R(k)}{\partial k} < 0 \).
Appendix G.1.2 Rush

Consider the effect of the opportunity function on the size of the rush of migration. By proposition 2 and equation (Appendix A.3) the rush of migration is completely characterized by the opportunity function. Intuitively, equation (Appendix A.3) demonstrates that the number of individuals that rush to create a new location is given by the point where the opportunity function intersects its average, which corresponds to the maximum of the average opportunities function. The point at which the average is maximized is unaffected by either a level shift or a proportional shift.

COMPARATIVE STATIC 2 The number of individuals that rush to create a new location depends only on the opportunities in the new location but is unaffected by (i) a level change in the opportunities in the new location or (ii) a proportional change in the opportunities in a location.

i) Example Two Revisited With Property Tax and Rush

Consider the case where a rush of migration creates city 2. To ensure the conditions in proposition 2 hold set $\gamma = 1$ and $\phi = 2$. If the city imposes a property tax, the opportunities in the new city decrease and the difference between opportunities becomes smaller. The number of individuals that rush to create the city is derived from equating the opportunity function with its average according to equation (Appendix A.3). For the example with $\gamma = 1$ and $\phi = 2$ this leads to a rush of migration of size,

$$J_1 = \frac{3v}{4m}$$

(Appendix G.2)
which is unaffected by the property tax. Thus, the number of individuals that rush to create a city is independent of the property tax.

**Appendix G.2 The Role of the Income Function with Implications for Income Taxation**

This section investigates how changes in the income function’s level and slope affect location growth. The effect of income taxation on location growth is demonstrated using comparative static 3. The effect of the income function on migration is found by taking the derivative of migration given in equation (9) with respect to the level $Y_2(N_2)$ and slope $Y_2'(N_2)$ of the income function.

**COMPARATIVE STATIC 3** The income produced within a location affects the location’s creation and growth such that (i) a proportional decrease (a flattening) in the income function causes the location’s growth rate to decrease, (ii) a level change in the income function does not affect its growth rate, and (iii) a level decrease in the income function causes the location to be created later in time.

1) Proportional Shift

The growth rate of a location decreases with a proportional decrease (a flattening) in the income function of a location. This result follows directly from equation (9) and contrasts with the result of a proportional decrease of the opportunity function. The intuition follows from the fact that the benefit individuals receive from income depends on two factors: the income function and the speed of migration. The present value of income in location 2 increases as the speed of migration increases because individuals spend less time with a lower level of income relative to location 1. When the difference in the income function decreases, the speed of migration must decrease such that the difference in the present value of income received over time remains the same.

2) Level Shift

The migration pattern in equation (9) does not depend on the income function in the new location, $Y_2$, implying that a level shift to the income function does not have an effect on the growth rate of the new location. The intuition is similar to a level shift of the opportunity function. Migration depends on differences of the income and opportunity functions such that the differences are equal to ensure there are no arbitrage opportunities. However, a level shift in the income function does have an effect on when the location is created. Intuitively, the location is created when the income and opportunities in the new location are enough to balance forgoing the income in the established location, which occurs later when there is a negative level shift to the income function in the new location.

3) Example Two Revisited With An Income Tax

Consider the effect of income taxation in the first example. Each city produces the

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24 The numerator and denominator in equation (9) are negative.
composite good $c$ as well as pollution which combine to produce income in the city. The income function with income taxation is given by,

$$Y_2(N) = (1 - \tau_I)(AN_2^\xi - BN_2^\beta)$$  (Appendix G.3)

where $\tau_I$ is the income tax rate imposed in the new city with production net pollution as its base.

To determine the effect of income taxation on city growth consider first the effect of an income tax within a city and then apply the result from comparative static 3. Income taxation in the new city has two effects: it lowers income within the city for a given population level and it decreases the difference between income produced with different levels of population. The effect of income taxation on the income function is depicted in Figure 7 where city B has a larger income tax than city A. Using the results in comparative static 3, income taxation causes cities to be created later in time and to grow slower. These effects are depicted in Figure 4 where the city with an income tax is given by the short-dashed line in contrast to the baseline growth path given by the solid line.

Figure 7: Differences in Average Production (City A Grows Faster Than City B)

Appendix H  Area and Distance for a Given Lot of Land

This appendix derives the area and distance of each lot of land in the first example given in the paper.

The city grows in a spiral around the central business district which uses $2\pi + 1$ parcels of land for production. For tractability the city grows according to a simple Archimedean spiral characterized by $r = \theta$, where $r$ is the radius, $\theta$ is the angle. Each parcel of land is
assumed to be formed by two lines radiating from the spiral’s pole. The angle between the two radiating lines is assumed to be constant and is denoted by $\bar{\theta} = 1$. The area of the parcel of land given to resident with rank $k$ is given by the following expression. 

$$
\nu(k) = \frac{1}{2} \int_{\bar{\theta} k}^{\bar{\theta} (k+1)} (\theta^2 - (\theta - 2\pi)^2) d\theta
$$

$$
= 2\pi \int_{\bar{\theta} k}^{\bar{\theta} (k+1)} (\theta - \pi) d\theta
$$

$$
= 2\pi(\theta^2/2 - \pi\theta)_{\bar{\theta} k}^{\bar{\theta} (k+1)}
$$

$$
= \bar{\theta}^2\pi 2k + \bar{\theta}^2\pi - 2\pi^2 \bar{\theta}
$$

This gives the area of the parcel of land for resident with rank $k$ in terms of $k$ and parameters.

Residents of a city must travel to the CBD to work. The expression below gives the distance of a resident’s commute, which is given by the shortest distance between their parcel of land and the pole of the spiral. The parcel of land for resident with rank $k$ is bounded by the spiral with edges that begin with radii $r = \bar{\theta}(k + 2\pi)$ and $r = \bar{\theta}(k + 2\pi) - 2\pi$. Therefore the closest distance is given by $k$ because $\bar{\theta}$ is assumed to be 1. Given that the city is formed by a rush, the number of movers that form the rush and the time $\tau$ will be uniquely determined. Each mover in the rush has rational expectations and expects to receive the average rank benefit. To ensure the rushing movers, $\{1, J^I\}$, do not have an incentive to deviate from rushing and migrate late at time $\tau^{\text{rush}} + dt$ the expected rank payoff must be greater than or equal to the rank payoff for mover $J^I + dk$. Similarly, to ensure that movers not in the rush do not have an incentive to migrate early with the rush the the rank payoff for the mover $J^I + dk$ must be greater than or equal to the expected rank payoff of the rush. Therefore the expected rank payoff of the rush must equal the rank payoff of the last rusher.

$$
\frac{1}{J^I} \int_0^{J^I} R(k) dk = R(J)
$$

To find the size of the rush $J$ first find the average rank payoff at point $J$,

$$
\frac{1}{J} \int_0^J R(k) dk = \frac{v}{\gamma + 1} J^\gamma - \frac{m}{\phi + 1} J^\phi.
$$

Set this equal to the opportunity of being mover $J$,

$$
\frac{v}{\gamma + 1} J^\gamma - \frac{m}{\phi + 1} J^\phi = vJ^\gamma - mJ^\phi,
$$

and rearrange to get,

$$
J = \left( \frac{\gamma v(\phi + 1)}{m\phi(\gamma + 1)} \right)^\frac{1}{\phi-\gamma}.
$$

25The area given between two curves, $r_1$ and $r_2$, in between the angles $a$ and $b$ is given by $1/2 \int_a^b r_1^2 - r_2^2 d\theta$. 

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With this closed form solution for the number of rushers it is possible to do comparative statics, to understand when rushes will be large and when rushes will small. Given the constraint that $\phi > \gamma$ the following comparative statics hold.

\[
\begin{align*}
\frac{\partial J}{\partial m} &< 0 \\
\frac{\partial J}{\partial v} &> 0 \\
\frac{\partial J}{\partial \phi} &< 0 \\
\frac{\partial J}{\partial \gamma} &> 0 
\end{align*}
\]

Substituting the example in the microfoundations section where $\gamma = 1$ and $\phi = 2$, the size of the rush is given by

\[ J = \frac{3v}{4m} \]

which is the equation provided in the paper.